

Some Exact Blowup Solutions to the Pressureless Euler Equations in R^N

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Abstract

The pressureless Euler equations can be used as simple models of cosmology or plasma physics. In this paper, we construct the exact solutions in non-radial symmetry to the pressureless Euler equations in R^N :

$$\left\{ \begin{array}{l} \rho(t, \vec{x}) = \frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)} \vec{x}, \\ a(t) = a_1 + a_2 t, \end{array} \right. \quad (1)$$

where the arbitrary function $f \geq 0$ and $f \in C^1$; $s \geq 1$, $a_1 > 0$ and a_2 are constants.

In particular, for $a_2 < 0$, the solutions blow up on the finite time $T = -a_1/a_2$.

Moreover, the functions (1) are also the solutions to the pressureless Navier-Stokes equations.

Key Words: Pressureless Gas, Euler Equations, Exact Solutions, Non-Radial Symmetry, Navier-Stokes Equations, Blowup, Free Boundary

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1 Introduction

The N -dimensional Euler equations can be formulated in the following form:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\ \rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \delta \nabla P(\rho) = 0, \end{cases} \quad (2)$$

with $\vec{x} = (x_1, x_2, \dots, x_N) \in R^N$ and As usual, $\rho = \rho(t, \vec{x})$ and $\vec{u}(t, \vec{x}) \in R^N$ are the density and the velocity respectively. If $\delta = 1$, the system is with pressure and $P = P(\rho)$ is the pressure. The γ -law on the pressure, i.e.

$$P(\rho) = K\rho^\gamma, \quad (3)$$

with $K > 0$, is a universal hypothesis. If $\delta = 0$, the system is pressureless:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\ \rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] = 0. \end{cases} \quad (4)$$

The system can be used as models of cosmology [20] or plasma physics [2]. There are also intensive studies for the pressureless Euler equations (4) in the recent literature [3], [4], [6] and [11]. The constant $\gamma = c_P/c_v \geq 1$, where c_p and c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats. In particular, the fluid is called isothermal if $\gamma = 1$. The Euler equations (2) govern the evolutionary phenomena of classical fluid dynamics. For the detailed studies of the Euler equations (2), see [1], [8] and [12].

For the Euler equations (2) in radial symmetry

$$\rho(t, x) = \rho(t, r) \text{ and } \vec{u} = \frac{\vec{x}}{r} V(t, r) := \frac{\vec{x}}{r} V, \quad (5)$$

with $r = \left(\sum_{i=1}^N x_i^2 \right)^{1/2}$,

there exists a family of solutions [9] for $\gamma > 1$,

$$\begin{cases} \rho(t, r) = \begin{cases} \frac{y(r/a(t))^{1/(\gamma-1)}}{a(t)^N}, & \text{for } y(\frac{r}{a(t)}) \geq 0; \\ 0, & \text{for } y(\frac{r}{a(t)}) < 0 \end{cases}, \quad V(t, r) = \frac{\dot{a}(t)}{a(t)} r, \\ \ddot{a}(t) = \frac{-\lambda}{a(t)^{1+N(\gamma-1)}}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \\ y(x) = \frac{(\gamma-1)\lambda}{2\gamma K} x^2 + \alpha^{\theta-1}, \end{cases} \quad (6)$$

[15] for $\gamma = 1$,

$$\left\{ \begin{array}{l} \rho(t, r) = \frac{e^{y(r/a(t))}}{a(t)^N}, u(t, r) = \frac{\dot{a}(t)}{a(t)} r \\ \ddot{a}(t) = \frac{-\lambda}{a(t)}, a(0) = a_0 > 0, \dot{a}(0) = a_1, \\ y(x) = \frac{\lambda}{2K} x^2 + \alpha, \end{array} \right. \quad (7)$$

where λ , α , a_0 and a_1 are constants.

The analytical solutions are found because of the techniques of separation method of self-similar solutions. The method were used to handle other similar systems in [5], [9], [10], [13], [14], [15], [16], [17], [18] and [19].

It is very natural to see that for the pressureless Euler equations (4), there exists a class of solutions

$$\rho(t, r) = \frac{f(r/a(t))}{a(t)^N}, \quad V(t, r) = \frac{\dot{a}(t)}{a(t)} r, \quad (8)$$

with the arbitrary function $f \geq 0$ and $f \in C^1$; and $a(t) > 0$ and $a(t) \in C^1$.

In this article, we have obtained the more general results about the pressureless Euler equations (4) in the following theorem:

Theorem 1 *For the N -dimensional pressureless Euler equations (4), there exists a family of solutions:*

$$\left\{ \begin{array}{l} \rho(t, \vec{x}) = \frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)} \vec{x}, \\ a(t) = a_1 + a_2 t, \end{array} \right. \quad (9)$$

with the arbitrary function $f \geq 0$ and $f \in C^1$; and $s \geq 1$, $a_1 > 0$ and a_2 are constants.

In particular, for $a_2 < 0$, the solutions blow up on the finite time $T = -a_1/a_2$.

2 Separation Method

Regards to the continuity equation of mass (4)₁, we found that the following solution structures fit it well:

Lemma 2 *For the equation of conservation of mass,*

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0, \quad (10)$$

there exist general solutions,

$$\rho(t, \vec{x}) = \frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N}, \quad \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)} \vec{x}, \quad (11)$$

with the arbitrary function $f \geq 0$ and $f \in C^1$; $a(t) > 0$ and $a(t) \in C^1$; and the constant $s \geq 1$.

Proof. We just plug (11) into (10). Then, we have:

$$\rho_t + \nabla \bullet \vec{u} \rho + \nabla \rho \bullet \vec{u} \quad (12)$$

$$= \frac{\partial}{\partial t} \left[\frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N} \right] + \left[\nabla \bullet \frac{\dot{a}(t)}{a(t)} \vec{x} \right] \frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N} + \nabla \frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N} \bullet \frac{\dot{a}(t)}{a(t)} \vec{x} \quad (13)$$

$$= \frac{-N\dot{a}(t)}{a(t)^{N+1}} f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right) + \frac{1}{a(t)^N} \frac{\partial}{\partial t} f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right) \quad (14)$$

$$+ \sum_{i=1}^N \frac{\dot{a}(t)}{a(t)} \frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N} + \sum_{i=1}^N \frac{\partial}{\partial x_i} \left[\frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N} \right] \frac{\dot{a}(t)}{a(t)} x_i \quad (15)$$

$$= \frac{-N\dot{a}(t)}{a(t)^{N+1}} f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right) - \frac{\dot{f}\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right) s \sum_{i=1}^N x_i^s \dot{a}(t)}{a(t)^N a(t)^{s+1}} \quad (16)$$

$$+ N \frac{\dot{a}(t)}{a(t)} \frac{f\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right)}{a(t)^N} + \sum_{i=1}^N \frac{\dot{f}\left(\frac{1}{a(t)^s} \sum_{i=1}^N x_i^s\right) s x_i^{s-1} \dot{a}(t)}{a(t)^N a(t)^s} x_i \quad (17)$$

$$= 0 \quad (18)$$

The proof is completed. ■

Remark 3 We notice that the novel lemma fully covers Lemma 3 in [15] for the mass equation (3) in radial symmetry,

$$\rho_t + \rho_r V + \rho V_r + \frac{N-1}{r} \rho V = 0 \quad (19)$$

which showed that there exists a class of solutions

$$\rho(t, r) = \frac{f(r/a(t))}{a(t)^N}, \quad V(t, r) = \frac{\dot{a}(t)}{a(t)} r, \quad (20)$$

with the arbitrary function $f \geq 0$ and $f \in C^1$; and $a(t) > 0$ and $a(t) \in C^1$.

The proof of Theorem 1 is similar to the ones in [5], [13] and [14]. The main idea is to put the exact solutions to check that if they satisfy the system (4) only.

Proof of Theorem 1. From the above lemma, it is very clear to verify that our solutions (9) satisfy the mass equation (4)₁. Regards to the pressureless momentum equation (4)₂, we have:

$$\rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] \quad (21)$$

$$= \rho \left[\left(\frac{\dot{a}(t)}{a(t)} \vec{x} \right)_t + \frac{\dot{a}(t)}{a(t)} \vec{x} \cdot \nabla \frac{\dot{a}(t)}{a(t)} \vec{x} \right] \quad (22)$$

$$= \rho \left[\left(\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) \vec{x} + \frac{\dot{a}^2(t)}{a^2(t)} \vec{x} \right] \quad (23)$$

$$= \rho \left[\frac{\ddot{a}(t)}{a(t)} \vec{x} \right] \quad (24)$$

$$= 0, \quad (25)$$

with

$$a(t) = a_1 + a_2 t. \quad (26)$$

The proof is completed. ■

Remark 4 In particular, the solutions (9) for $s = 1$,

$$\begin{cases} \rho(t, \vec{x}) = \frac{f\left(\frac{1}{a(t)} \sum_{i=1}^N x_i\right)}{a(t)^N}, \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)} \vec{x}, \\ a(t) = a_1 + a_2 t, \end{cases} \quad (27)$$

are line sources or sinks. For the physical significance of such kind of solutions, the interested readers may refer P.409-410 of [7] for details.

On the other hand, another family of solutions are provided here.

Theorem 5 Denote $A\vec{x} := (\frac{x_1}{\bar{u}_{01}}, \frac{x_2}{\bar{u}_{02}}, \dots, \frac{x_i}{\bar{u}_{0i}}, \dots, \frac{x_N}{\bar{u}_{0N}})$ for all $\bar{u}_{0i} \neq 0$ and $A\vec{x} := (\frac{x_1}{\bar{u}_{01}}, \frac{x_2}{\bar{u}_{02}}, \dots, 0 \cdot x_i, \frac{x_N}{\bar{u}_{0N}})$, for some $\bar{u}_{0i} = 0$. For the pressureless Euler equations (4), there exists a family of solutions:

$$\rho = f(A\vec{x} - t), \quad \vec{u} = \vec{u}_0, \quad (28)$$

with the arbitrary function $f \geq 0$ and $f \in C^1$; and $\vec{u}_0 = (\bar{u}_{01}, \bar{u}_{02}, \dots, \bar{u}_{0N}) \in R^N$ is a constant vector.

It is trivial to check that the above theorem is true. We skip the proof here.

Remark 6 *The functions (9) and (28) are also the solutions to the pressureless Navier-Stokes equations,*

$$\left\{ \begin{array}{l} \rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\ \rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] = \mu \nabla (\nabla \cdot \vec{u}), \end{array} \right. \quad (29)$$

$\mu > 0$ is a constant.

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